# Plan of Activities and Research Project of Rafael O. Ruggiero for a 3 month stage at Universities of Aix Marseille and Avignon, from 01-01-2017 to 03-30-2017

### Conservative dynamics and its connections with global geometry and topology

Rio de Janeiro, July 13th, 2016

## Introduction

The research project considers problems in five main research areas: Anosov and expansive geodesic flows, the theory of Lagrangian graphs, rigidity problems in Finsler geometry, genericity from Mañé's viewpoint, and phase transition of expansive, nonhyperbolic homeomorphisms in surfaces. Most of my research partners are professors in France: Professors Patrick Foulon (CIRM), Ludovic Rifford (Nice), Renaud Leplaideur (Brest), and I have many common research interests with Professors Michel Boileau (Marseille), Marie Claude Arnaud (Avignon) and Thierry Barbot (Avignon).

## 0.1 Program of Activities

Let us give a schedule of the main scientific activities to be implemented during the stage.

1. From January 1rst to February 28th, stay in the University of Aix Marseille to work with Professors Michel Boileau and Patrick Foulon.

2. From March 1rst to March 30th, stay in the University of Avignon to work with Professors Marie-Claude Arnaud and Thierry Barbot. Minicourses about Lagrangian invariant tori, manifolds without conjugate points, Finsler geometry and geometric control theory applied to generic theory of Hamiltonians, are planned during the visit.

3. Participation in seminars, workshops and all research activities of the Universities of Aix Marseille, CIRM, Avignon, during the whole visit.

## 0.2 Detailed Research Project

We subdivide the research project in 5 main streams to explain problems and their backgrounds clearer.

1. GEODESIC FLOWS IN MANIFOLDS WITHOUT CONJUGATE POINTS.

Our main research subject is the following:

**The topology of the set of metrics without conjugate points in a compact manifold ($C^2$ closure and boundary, conectivity, etc)**.

The motivation for the subject is the work of R. Ruggiero [R2]: The $C^2$-interior of the set of Riemannian metrics without conjugate points in a compact manifold is the set of Anosov metrics. This leads us to the following natural question: Is every metric without conjugate points in a compact surface with genus $\geq 2$ accumulated in the $C^2$ topology by Anosov metrics? It is not difficult to get examples of non-Anosov surfaces with genus $\geq 2$ which are in the closure of Anosov metrics, but a complete characterization of the $C^2$-closure of Anosov metrics seems to be very hard. The result recently obtained by Jane-Ruggiero [JR] sheds some light on the subject: Every compact surface without focal points and genus $\geq 2$ where the region of positive curvature is a finite union of "bubbles" is in the closure of Anosov metrics. The proof of this fact involves a new technique in the study of the boundary of stability of geodesic flows: the Ricci-Yang-Mills flow, a sort of magnetic version of the Ricci flow. From this work arises an interesting question: Does the conformal class of a compact surface without conjugate points and genus $\geq 2$ is arc connected? Notice that the conformal class of a compact surface of genus $\geq 2$ and non-positive curvature is arc connected, since the Ricci flow "pushes" conformally such metric into the set of metrics of negative curvature, and then Hamilton shows that there exists a path of conformal metrics of negative curvature joining the surface to a hyperbolic surface (see [JR] for instance for details).

2. LAGRANGIAN INVARIANT TORI.

We focus on two main problems.

1. **Birkhoff Theorems (actually conjectures) for Lagrangian, invariant tori of Hamiltonian flows. Namely, to characterize Lagrangian tori, invariant by a Hamiltonian flow in $T^n$, which are graphs of the canonical projection.**
2. **To find the finest topology where generically there is no Lagrangian invariant graph in a high energy level of a Tonelli Hamiltonian.**

The first item is motivated by the works of Bialy and Polterovich (see for instance [BP], [Pol]) and more recently Marie-Claude Arnaud (see [Arnaud]). Applying some of the ideas of Arnold to study singularities of Lagrangian submanifolds (notably the Maslov index) and Aubry-Mather theory, these authors obtain partial extensions of Birkhoff theorems for invariant curves of area preserving twist maps of the annulus to Lagrangian invariant tori. In [CR1] we extend some of the results obtained by Bialy-Polterovich in two dimensions. This showed that some of the assumptions on their works were somehow technical and not quite essential.

Regarding the second problem, we look for the largest $\beta \in (0,1)$ such that given a supercritical energy level of a Hamiltonian, a $C^{1,\beta}$-perturbation of the Hamiltonian has no lagrangian invariant graph in the same level. The interest of this problem arises from KAM theory, which grants the persistence of Lagrangian invariant tori by $C^{2,\beta}$ perturbations of totally integrable Hamiltonians. However, KAM theory does not hold in weaker topologies. Herman [Herman] noticed that there exist $C^{d+1-\beta}$ perturbations of a Hamiltonian in $T^d$, com $\beta \in (0,1)$, without $C^1$ Lagrangian invariant graphs. This does not imply the non-existence of Lagrangian invariant graphs since such graphs are in general Lipschitz.

An idea due to Bangert [Ban1] show that there exist metrics in $T^2$ $C^1$-arbitrarilly close to a flat metric with the property of having no Lagrangian invariant graphs. This fact is a counterpart of know properties of area preserving twist maps of the annulus (see Takens [Takens], Mather [Mather]). Other references in the literature [R8], [R11], [R13], show the $C^1$-density of non-existence of Lagrangian invariant graphs in all supercritical levels of a mechanical Lagrangian in $T^2$. Recently, in a joint work with R. Pacheco, we show that given a reversible Finsler two torus there exists $\beta > 0$ and a $C^{1,\beta}$-perturbation such that the perturbed geodesic flow has no Lagrangian invariant graphs. We think that the reversibility assumption is not essential for the result.

3. RIGIDITY IN FINSLER GEOMETRY.

We consider the following problems.

1. **A Finsler, k-basic, $C^\infty$, compact surface of genus $\geq 2$ is Riemannian.**

2. **A Finsler k-basic, $C^\infty$ two torus without conjugate points is flat.**

The first problem arises from the work of G. Paternain: Finsler k-basic compact, analytic surfaces of genus $\geq 2$ are Riemannian. We would like to point out that the Hamiltonian flow in an supercritical energy level is the geodesic flow of a Finsler metric, so the dynamics of these flows is equivalent to Hamiltonian dynamics in high energy levels. Many partial results [BR2], [BR3], [BR4], [BR5] indicate that the answer to the first problem should be positive. Recently, in a joint work with Professor Patrick Foulon of CIRM, Marseille, we show that the geodesic flow of every compact Finsler k-basic surface has a first integral, which does not vanish on the subset of the unit tangent bundle where the Finsler metric is not Riemannian (namely, where the Cartan tensor does not vanish). Applying this result and the main theorem in [BR6] we show the first statement is true assuming additionally that the surface has no conjugate points. The existence of a first integral for the geodesic flow of Finsler k-basic compact surfaces extends known results in the context of Randers metrics of constant curvature, Katok-Ziller spheres are included in this category.

The second problem is a Finsler version of the Hopf conjecture, that is false in general due to well known examples by Busemann but that holds under the additional assumptions of k-basic and analyticity according to [BCR].

## 4. GENERICITY FROM MAÑÉ'S VIEWPOINT, CONFORMAL GEOMETRY AND CONTROL THEORY.

We apply the notion of genericity proposed by R. Mañé [Ma2] to study generic properties of Aubry-Mather measures to study generic properties of Tonelli Lagrangians. A given property $P$ of a Tonelli Lagrangian $L$ is $C^k$ Mañé generic if it holds for perturbations of $L$ of the form $L+U$, where $U$ is a scalar function (or potential). For mechanical Lagrangians, Maupertuis principle says that such perturbations are all conformal to the metric defining the Lagrangian. Recent results applying geometric control theory with Professor L. Rifford of Nice University [RR] show that $C^1$-Mañé's genericity of Poincaré maps of closed orbits of Lagrangians is equivalent to full $C^1$-genericity. In a joint work with L. Rifford e A. Lazrag [LRR] we obtain a Franks Lemma from Mañé's viewpoint with many applications to persistent geodesic flows. One of the main advantages of Mañé's point of view of genericity is the simplicity of the expression of the differential of the perturbed Hamiltonian flow in terms of the unperturbed one, giving a cleaner, easier way to understand genericity when compared with the classical literature about the subject. Our project is to continue the study of genericity of Hamiltonian dynamics from this point of view, in particular the generic theory in special families of Hamiltonians (geodesic, magnetic for instance). We would like to understand possible applications of the techniques of generic conformal perturbations to investigate the relationship between: (a) persistent dynamics, (b)density of ergodicity (in the spirit of Donnay-Pugh); with non-vanishing Lyapunov exponents.

## 5. PHASE TRANSITION (OR NOT?) OF HIGHER DIMENSIONAL MANNEVILLE-POMMEAU DYNAMICS.

In a joing research project with Professor Renaud Leplaideur of the Université de Bretagne Occidental we study examples of conservative smooth homeomorphisms of the torus which are expansive, non-Anosov, having an indifferent fixed point with stable and unstable manifolds where the dynamics is, respectively, of stable - unstable Manneville-Pomeau type. The "unstable" Manneville-Pomeau map $T(x) = x + x^{1+s}$ of the interval $[-1, 1]$ has been decisive to understand the phase transition of physical measures under changes of the parameter $s$, $s > 0$. Perhaps the most famous piece of work about the subject is due to L. S. Young. Higher dimensional, expansive, conservative examples with invariant submanifolds where the dynamics is of Manneville-Pomeau type are quite few. The first time, as far as we know, that such examples appear in the literature is in a paper by Katok [Katok] with parameter $s < \frac{1}{2}$. However, since the phase transition in the one-dimensional case occurs at $s = 1$ this family is not enough to our purposes. Our goal is **to exhibit a family that complements Katok's family of maps, in order to study what happens at $s = 1$.** Based on some examples of expansive geodesic flows with non-positive curvature having indifferent closed geodesics our conjecture is that there is no phase transition. We also plan to obtain an estimate of decay of correlations (mixing rate) to extend some observations made by Liverani-Martens [LM] in the case of expansive dynamics with an indifferent fixed point where invariant sets are tangent of cubic order.

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