Reduction algebras and quantum groups

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Background. Mickelsson algebras (or reduction algebras) were introduced in [M] for the study of Harish-Chandra modules of reductive groups. The Mickelsson algebra, related to a real reductive group G, acts on the space of highest weight vectors of its maximal compact subgroup, and each irreducible Harish-Chandra module of the initial reductive group is uniquely characterized by this action. It can be realized as certain sub-quotient of the universal enveloping algebra of the Lie algebra of G. A similar construction can be given for any associative algebra A, which contains a universal enveloping algebra U(g) (or its q-analog) of a contragredient Lie algebra g with a fixed Gauss decomposition.

Previous work. This collaboration has already led to publication of five articles in mathematics/mathematical physics (the last one was submitted in October 2015).

In the paper [KO], generalizing the ideas of Zhelobenko [Z], we studied in details the algebraic structure of Mickelsson algebras. In particular, we described a family of automorphisms for a wide class of Mickelsson algebras. They form a representation of the corresponding braid group by automorphisms and transform the Cartan elements by means of the shifted Weyl group action. These results were later applied to the description of the structure constants of diagonal reduction algebras, for the generalization of Harish-Chandra isomorphism and to the representation theory of Yangians [KO2,KO3,KNV,KN]. A particular case of reduction algebras is known under the name of dynamical quantum groups, see e.g. [ES]. They are fruitfully exploited for the study of quantum integrable models.

In the paper [KO4] we continued the general study of the structure of reduction algebras. We have given several proofs of the Poincaré–Birkhoff–Witt theorem for reduction algebras, showed that reduction algebras admit fields of fractions, proposed Gelfand–Kirillov type conjectures for these fields and verified the conjectures in the simplest cases.

Research plans. One of our goals is to establish a close connection between Mickelsson algebras and quantum groups. This includes: from one side a Yang-Baxter type description of various Mickelsson algebras and the use of quantum group techniques for their algebraic description; from another side the application of the methods developed in our investigations of Mickelsson algebras to dynamical quantum groups and related integrable models.

Another goal is an independent study of the representation theory of Mickelsson algebras with applications to classical problems in the theory of Lie groups and in mathematical physics. The first problem is to interpret Clebsch-Gordan coefficients as matrix coefficients of representations of diagonal reduction algebras, studied in [KO2,KO3]. The second problem is an explicit description of representations of discrete series for non-compact real Lie groups. This question can be reduced to a consecutive construction of reduction algebras for Mickelsson algebras itself. Here we can use the construction of the extremal projector [T] and Shapovalov form for Mickelsson algebras [KN], but the crucial difficulty is an analytical study of singularities, appearing in the construction of representations. Further we also plan to pass to affine Lie algebras and to apply the technique of reduction algebras for the study of the so called coset models [FMS] in conformal field theory.

Zhelobenko automorphisms may be also regarded as analogs of Knapp-Stein intertwining operators [KS] for the category O of highest weight modules of a semi-simple Lie algebra. However their constructions are quite different; the first operators are essentially differential operators while the second ones are integral operators. We intend to work out the unified analytical theory for all of them, so that the rational coefficients of the matrix elements of Zhelobenko maps become degenerations of the Harish-Chandra c-function.

The diagonal reduction algebras deserve a separate treatment. Morally, the representation theory of a diagonal reduction algebra $\mathcal{D}\mathbf{g}$ of a reductive Lie algebra $\mathbf{g}$ is responsible for the decompositions of tensor products of $\mathbf{g}$-modules. We plan to give a description of $\mathcal{D}\mathbf{g}$ in terms of the reflection equation involving the dynamical R-matrix for $\mathbf{g}$. It would then follow that $\mathcal{D}\mathbf{g}$ possesses a bialgebra structure. An important step in this description should rely on the theory of dynamical differential operators which is worked out by O. Ogievetsky and his PhD student B. Herlemont.

An experimental evidence shows that there exists a "dynamization" functor which allows to transport information about $\mathbf{g}$-modules to that about $\mathcal{D}\mathbf{g}$-modules. We are aiming at providing an effective theory of this construction. A recent development of the principal series representations for the Lie algebras $\mathbf{gl}_n$ is a factorization of the L-operator. The dynamization functor should allow to discover an analogous factorization for $\mathcal{D}\mathbf{gl}_n$.

We plan, with our PhD students, to investigate these phenomena for Lie algebras of other classical types and their q-deformations.

During his visit, Serguei Khoroshkin will give a seminar on subjects related to modern mathematical physics, in particular, related to the present project.

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